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The three-dimensional laminar boundary layer on a flat plate

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A series expansion is derived for the three-dimensional boundary-layer flow over a flat plate, arising from a general main-stream flow over the plate. The series involved are calculated as far as terms of order ξ^2, where ξ is a non-dimensional parameter defining distance measured from the leading edge of the plate. The results are applied to an example in which the main stream arises from the disturbance of a uniform stream by a circular cylinder mounted downstream from the leading edge of the plate, the axis of the cylinder being normal to the plate. Calculations are made for shear stress components on the plate, and for the deviation of direction of the limiting streamlines from those in the main stream.

1. Introduction

The object of this investigation is the determination of the leading terms in an expansion in series for the laminar boundary layer near the leading edge of a flat plate, and such that the expansion is sufficiently general to correspond to a wide class of possible flows in the main stream over the plate.

The study of three-dimensional boundary layers on flat plates is greatly simplified by the absence of geometrical complications, and Blasius-type solutions of the equations have been derived by Hansen & Herzig (1956). Previously, both Loos (1955) and Sowerby (1954) had discussed a special case of such solutions. These solutions relate to boundary layers associated with a special class of main-stream flows—namely, the class in which the streamlines form a system of translates. Nevertheless, they exhibit genuine three-dimensional effects, such as the divergence of the direction of limiting streamlines from the direction of the external streamlines, and the work of Hansen & Herzig has been used by Cooke (1959) as a test for the accuracy of his approximate solutions. They are also exact solutions of boundary-layer equations, in the sense that the Blasius function is an exact solution of the two-dimensional boundary-layer equations. The solution given in this paper is more restricted in that it is an expansion in series, of which only the first few terms are derived. It corresponds, however, to realistic main-stream distributions, and serves to provide detailed information in the early stages of development of the boundary layer on a flat plate.

2. Boundary-layer equations, and transformation of co-ordinates

Let $O(x, y, z)$ be a system of rectangular Cartesian co-ordinates, with the plate situated in the half-plane $z = 0, x \geqslant 0$. Then, if u, v, w are appropriate components of the velocity $\mathbf{v}$ of the fluid, for steady flow the boundary-layer equations in a usual notation are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2}, \tag{2.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\frac{\partial^2 v}{\partial z^2}, \tag{2.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{2.3}$$

Since $\partial p/\partial z = 0$, the pressure p is determined by the inviscid flow in the main stream; thus, if U, V, W are velocity components for this flow,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}, \tag{2.4}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}, \tag{2.5}$$

where now in these last relations the terms on the right are evaluated setting $z = 0$, so that U, V are treated as functions of x, y only.

The curved main-stream flow over the plate may be considered as established by some disturbing body in a uniform main stream, such as a cylinder set with its axis normal to the plate; this is the example which has been chosen to illustrate an application of the general results. Complications due to the boundary-layer and wake effects associated with the cylinder may be avoided by placing the cylinder downstream from the leading edge of the plate, so that there exists a region of inviscid main-stream flow near the leading edge of the plate. Thus a representative length 'a' may be selected (in this instance the radius of the cylinder), and the following non-dimensional variables may be formed:

$$\xi = x/a, \quad s = y/a, \quad \eta = (U/2\nu a\xi)^{\frac{1}{2}} z. \tag{2.6}$$

The velocity components U and V are functions of ξ, s only, and the velocity components in the boundary layer are expressed as

$$u = U\,\partial f(\xi,\eta,s)/\partial\eta, \quad v = V\,\partial g(\xi,\eta,s)/\partial\eta \quad \text{and} \quad w = (U\nu/2a\xi)^{\frac{1}{2}} h(\xi,\eta,s). \tag{2.7}$$

An alternative but more complicated approach here might be the use of an extension of Görtler's transformation for the two-dimensional case, with the possibility then of including other three-dimensional boundary-layer flows in addition to the flow over a flat plate. The above transformation is simply the three-dimensional equivalent of Falkner's transformation; for both transformations see, for example, Rosenhead (1963, Ch. VI).

Substitution in the equation of continuity (2.3), and one integration with respect to η leads to the result (apart from an arbitrary function of ξ and s)

$$h = \eta f_\eta - f - \xi\left\{\frac{U_\xi}{U}(\eta f_\eta + f) + 2f_\xi + 2\frac{V_s}{U}g + 2\frac{V}{U}g_s + \frac{VU_s}{U^2}(\eta g_\eta - g)\right\}, \qquad (2.8)$$

in which a literal suffix denotes, as usual, differentiation with respect to the appropriate variable.

The boundary conditions on the functions f, g, h are clearly

$$f_\eta = g_\eta = h = 0, \quad \text{when} \quad \eta = 0,$$
$$f_\eta, g_\eta \to 1, \quad \text{as} \quad \eta \to \infty.$$

The boundary condition on h is evidently satisfied provided f and g satisfy also the conditions

$$f = g = f_\xi = g_s = 0, \quad \text{when} \quad \eta = 0.$$

Thus a complete set of boundary conditions for f and g is

$$\left.\begin{aligned} &f = f_\eta = f_\xi = g = g_\eta = g_s = 0, \quad \text{when} \quad \eta = 0,\\ &f_\eta, g_\eta \to 1, \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \qquad (2.9)$$

The coefficients arising from combinations of U, V and their derivatives, which occur in result (2.8), also arise in the transformed equations of motion, and it is convenient here to define these combinations as new functions of ξ and s.

Put

$$A = V/U, \quad B = U_\xi/U, \quad C = V_s/U, \quad D = VU_s/U^2, \quad E = V_\xi/V. \qquad (2.10)$$

Certain combinations of these functions occur later, and may be stated here. These are

$$\left.\begin{aligned} F &= B + 2C - D,\\ G &= 2(B+D),\\ K &= 2(C+E-B-D). \end{aligned}\right\} \qquad (2.11)$$

It is assumed that U and V are non-zero, at least in some region extending downstream from the edge of the plate. The condition that U should be non-zero (and also positive, incidentally) is the same as in the case of the two-dimensional Blasius boundary layer, but there is no physical reason why V should not assume zero values along some curve or straight line. It is assumed in the analysis to follow that V is non-zero at the edge of the plate; the alternative, namely $V(0, s) = 0$, will be discussed later, where it will be shown that this apparently exceptional case is indeed covered by the general results.

With regard to relations (2.10) it is evident that for a two-dimensional main stream in planes parallel to the plate the functions $A, \ldots, E$ reduce to four in number, since the equation of continuity is

$$\partial U/\partial x + \partial V/\partial y = 0,$$

and hence

$$B = -C. \qquad (2.12)$$

In general, however, this relation is not valid. For a three-dimensional main stream (such as, for example, a uniform stream disturbed by a hemisphere placed with its base on the plate) the equation of continuity is

$$\partial U/\partial x + \partial V/\partial y + \partial W/\partial z = 0,$$

in which the term $\partial W/\partial z$ is not zero in general.

The analysis in the next section is based on the general case, but it will be seen that, in the circumstances in which relation (2.12) is valid, some reduction would be possible in the number of functions f_{ij}, g_{ij} which are required to determine the velocity distribution in the boundary layer.

It remains now to state the transformed form of equations (2.1) and (2.2). With use of result (2.8), and the expressions for the pressure gradients, these equations are, respectively,

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \xi\{2f_\xi f_{\eta\eta} - 2f_\eta f_{\xi\eta} + B(ff_{\eta\eta} - 2f_\eta^2 + 2) + 2Cgf_{\eta\eta} - D(gf_{\eta\eta} + 2f_\eta g_\eta - 2) + 2A(g_s f_{\eta\eta} - g_\eta f_{\eta s})\} = 0, \quad (2.13)$$

$$g_{\eta\eta\eta} + fg_{\eta\eta} + \xi\{2f_\xi g_{\eta\eta} - 2f_\eta g_{\xi\eta} + Bfg_{\eta\eta} + 2C(gg_{\eta\eta} + 1 - g_\eta^2) - Dgg_{\eta\eta} + 2E(1 - f_\eta g_\eta) + 2A(g_s g_{\eta\eta} - g_\eta g_{\eta s})\} = 0. \quad (2.14)$$

3. Expansions in series

Equations (2.13) and (2.14), subject to the boundary conditions (2.9) may be solved by expansions in series of powers of ξ. It is assumed that the coefficients A etc. are analytic functions of ξ, so that

$$A = \sum_{n=0}^{\infty} \xi^n A_n, \quad (3.1)$$

where the A_n† are functions of s alone. Similar expansions hold for the remaining coefficients, and the expansions for the velocity functions are

$$f = \sum_{n=0}^{\infty} \xi^n f_n, \quad g = \sum_{n=0}^{\infty} \xi^n g_n, \quad (3.2)$$

where f_n, g_n are functions of both η and s. These functions, in fact, must later be decomposed further into a sum of functions of η alone, with coefficients in functions of s.

The boundary conditions on f_n, g_n are, from (2.9),

$$\left.\begin{aligned} f_n = \partial f_n/\partial\eta = g_n = \partial g_n/\partial\eta = \partial g_n/\partial s = 0, &\quad \text{when} \quad \eta = 0, \\ \partial f_0/\partial\eta,\ \partial g_0/\partial\eta \to 1, &\quad \text{as} \quad \eta \to \infty, \\ \partial f_n/\partial\eta,\ \partial g_n/\partial\eta \to 0, &\quad \text{as} \quad \eta \to \infty, \quad \text{for} \quad n \geq 1. \end{aligned}\right\} \quad (3.3)$$

The expansions above are substituted in equations (2.13) and (2.14) and coefficients of the various powers of ξ are equated to zero. The terms independent of ξ yield at once

$$\partial^3 f_0/\partial\eta^3 + f_0 \partial^2 f_0/\partial\eta^2 = 0, \quad (3.4)$$

$$\partial^3 g_0/\partial\eta^3 + f_0 \partial^2 g_0/\partial\eta^2 = 0, \quad (3.5)$$

and in view of the boundary conditions on f_0 and g_0 these functions are clearly independent of s and are equal, each being identical with the Blasius function for

† In this section, literal suffixes do not refer to partial differentiation.

two-dimensional flow. This is to be expected, since these results assert that close to the edge of the plate the boundary-layer flow is the Blasius flow as determined by local main-stream conditions.

For the powers of ξ up to ξ^2, and with use of the equality of f_0 and g_0, the following equations are obtained:

$$\frac{\partial^3 f_1}{\partial\eta^3}+f_0\frac{\partial^2 f_1}{\partial\eta^2}-2\frac{\partial f_0}{\partial\eta}\frac{\partial f_1}{\partial\eta}+3\frac{\partial^2 f_0}{\partial\eta^2}f_1+F_0 f_0\frac{\partial^2 f_0}{\partial\eta^2}+G_0\left\{1-\left[\frac{\partial f_0}{\partial\eta}\right]^2\right\}=0, \tag{3.6}$$

$$\frac{\partial^3 g_1}{\partial\eta^3}+f_0\frac{\partial^2 g_1}{\partial\eta^2}-2\frac{\partial f_0}{\partial\eta}\frac{\partial g_1}{\partial\eta}+3\frac{\partial^2 f_0}{\partial\eta^2}f_1+F_0 f_0\frac{\partial^2 f_0}{\partial\eta^2}+2(C_0+E_0)\left\{1-\left[\frac{\partial f_0}{\partial\eta}\right]^2\right\}=0, \tag{3.7}$$

$$\begin{aligned}\frac{\partial^3 f_2}{\partial\eta^3}+f_0\frac{\partial^2 f_2}{\partial\eta^2}-4\frac{\partial f_0}{\partial\eta}\frac{\partial f_2}{\partial\eta}+5\frac{\partial^2 f_0}{\partial\eta^2}f_2+3f_1\frac{\partial^2 f_1}{\partial\eta^2}-2\left[\frac{\partial f_1}{\partial\eta}\right]^2+F_1 f_0\frac{\partial^2 f_0}{\partial\eta^2}+G_1\left\{1-\left[\frac{\partial f_0}{\partial\eta}\right]^2\right\}\\ +F_0 f_0\frac{\partial^2 f_1}{\partial\eta^2}-2\frac{\partial f_0}{\partial\eta}\left\{(2B_0+D_0)\frac{\partial f_1}{\partial\eta}+A_0\frac{\partial^2 f_1}{\partial s\,\partial\eta}+D_0\frac{\partial g_1}{\partial\eta}\right\}\\ +\frac{\partial^2 f_0}{\partial\eta^2}\left\{B_0 f_1+(2C_0-D_0)g_1+2A_0\frac{\partial g_1}{\partial s}\right\}=0,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{\partial^3 g_2}{\partial\eta^3}+f_0\frac{\partial^2 g_2}{\partial\eta^2}-4\frac{\partial f_0}{\partial\eta}\frac{\partial g_2}{\partial\eta}+5\frac{\partial^2 f_0}{\partial\eta^2}f_2+3f_1\frac{\partial^2 g_1}{\partial\eta^2}-2\frac{\partial g_1}{\partial\eta}\frac{\partial f_1}{\partial\eta}+F_1 f_0\frac{\partial^2 f_0}{\partial\eta^2}\\ +2(C_1+E_1)\left\{1-\left[\frac{\partial f_0}{\partial\eta}\right]^2\right\}+F_0 f_0\frac{\partial^2 g_1}{\partial\eta^2}-2\frac{\partial f_0}{\partial\eta}\left\{(2C_0+E_0)\frac{\partial g_1}{\partial\eta}+A_0\frac{\partial^2 g_1}{\partial s\,\partial\eta}+E_0\frac{\partial f_1}{\partial\eta}\right\}\\ +\frac{\partial^2 f_0}{\partial\eta^2}\left\{B_0 f_1+(2C_0-D_0)g_1+2A_0\frac{\partial g_1}{\partial s}\right\}=0.\end{aligned} \tag{3.9}$$

From the form of equations (3.6)–(3.9), and the fact that f_0 is a function of η alone, it is evident that f_1 etc. can be expressed as linear combinations of functions of η alone, with coefficients in functions of s. Fortunately, the similarity between the equations for f_n and g_n leads to some simplification in the expressions which are required for g_n once the expressions for f_n have been decided.

A prime is now used to denote differentiation of a function of one variable with respect to that variable, and the expressions for f_1, g_1 are

$$\left.\begin{aligned} f_1 &= F_0 f_{11}(\eta)+G_0 f_{12}(\eta),\\ g_1 &= f_1+K_0 g_{11}(\eta),\end{aligned}\right\} \tag{3.10}$$

where f_{11}, f_{12}, g_{11} satisfy respectively the ordinary differential equations

$$\left.\begin{aligned}(L_1+3f_0'')f_{11} &= -f_0 f_0'',\\ (L_1+3f_0'')f_{12} &= (f_0')^2-1,\\ \text{and}\qquad L_1 g_{11} &= (f_0')^2-1,\end{aligned}\right\} \tag{3.11}$$

in which the differential operator L_1 is defined by

$$L_1 \equiv \frac{d^3}{d\eta^3}+f_0\frac{d^2}{d\eta^2}-2f_0'\frac{d}{d\eta}.$$

The corresponding results for f_2, g_2 are

$$\left.\begin{aligned} f_2 &= F_0^2 f_{21}(\eta) + F_0 G_0 f_{22}(\eta) + G_0^2 f_{23}(\eta) + F_1 f_{24}(\eta) + G_1 f_{25}(\eta) + \{K_0(2C_0 - D_0) \\ &\quad + 2A_0 K_0'\} f_{26}(\eta) + 2A_0 F_0' f_{27}(\eta) + 2A_0 G_0' f_{28}(\eta) + 2D_0 K_0 f_{29}(\eta), \\ g_2 &= f_2 + F_0 K_0 g_{21}(\eta) + G_0 K_0 g_{22}(\eta) + K_1 g_{23}(\eta) \\ &\quad + 2\{(2C_0 - D_0 + E_0) K_0 + A_0 K_0'\} g_{24}(\eta), \end{aligned}\right\} \tag{3.12}$$

and the differential equations satisfied by f_{2j}, g_{2j} are

$$\left.\begin{aligned} (L_2 + 5f_0'')f_{21} &= -3f_{11}f_{11}''' + 2(f_{11}')^2 - f_0''f_{11} - f_0 f_{11}''', \\ (L_2 + 5f_0'')f_{22} &= -3f_{11}f_{12}''' - 3f_{11}'''f_{12} + 4f_{11}'f_{12}' - f_0''f_{12} - f_0 f_{12}''' + 2f_0'f_{11}', \\ (L_2 + 5f_0'')f_{23} &= -3f_{12}f_{12}''' + 2(f_{12}')^2 + 2f_0'f_{12}', \\ (L_2 + 5f_0'')f_{24} &= -f_0 f_0'', \\ (L_2 + 5f_0'')f_{25} &= (f_0')^2 - 1, \\ (L_2 + 5f_0'')f_{26} &= -f_0'' g_{11}, \\ (L_2 + 5f_0'')f_{27} &= f_0'f_{11}' - f_0''f_{11}, \\ (L_2 + 5f_0'')f_{28} &= f_0'f_{12}' - f_0''f_{12}, \\ (L_2 + 5f_0'')f_{29} &= f_0' g_{11}', \\ L_2 g_{21} &= 2f_{11}'g_{11}' + 2f_0'f_{11}' - 3f_{11}g_{11}'' - f_0 g_{11}'', \\ L_2 g_{22} &= 2f_{12}'g_{11}' + 2f_0'f_{12}' - 3f_{12}g_{11}'', \\ L_2 g_{23} &= (f_0')^2 - 1, \\ L_2 g_{24} &= f_0' g_{11}', \end{aligned}\right\} \tag{3.13}$$

where
$$L_2 \equiv \frac{d^3}{d\eta^3} + f_0 \frac{d^2}{d\eta^2} - 4f_0' \frac{d}{d\eta}.$$

The boundary conditions on these functions are

$$\left.\begin{aligned} f_{ij}(0) &= f_{ij}'(0) = f_{ij}'(\infty) = 0, \\ g_{ij}(0) &= g_{ij}'(0) = g_{ij}'(\infty) = 0, \end{aligned}\right\} \tag{3.14}$$

since the boundary conditions (3.3) evidently then are satisfied, bearing in mind the previous results for f_0 and g_0.

In the general case considered above, therefore, it appears that the number of functions of η which must be determined to evaluate the flow to terms of order ξ^2 is no less than seventeen, and it is evident that this number would be greatly increased at the next stage. In the case of the two-dimensional main stream in which condition (2.12) is valid, the corresponding number of functions required for evaluation to order ξ^2 is eleven.

4. Modified results for the case $V(0, s) = 0$.

The exceptional case mentioned in §2 is considered here. The component of main-stream velocity V now has the form

$$V = \xi^r V^*(\xi, s),$$

where r is a positive integer and V^* is an analytic function of ξ, with $V^*(0, s) \neq 0$. Hence E is not analytic at $\xi = 0$, since $E = V_\xi / V$.

Re-define the functions (2.10) so that they are based on U and V^*; thus

$$A = V^*/U, \text{ etc.}$$

The expression for v is still

$$v = V\partial g(\xi, \eta, s)/\partial\eta.$$

With the above changes, equations (2.13) and (2.14) become

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \xi\{2f_\xi f_{\eta\eta} - 2f_\eta f_{\xi\eta} + B(ff_{\eta\eta} - 2f_\eta^2 + 2) + 2C\xi^r gf_{\eta\eta} - D\xi^r(gf_{\eta\eta} + 2f_\eta g_\eta - 2) + 2A\xi^r(g_s f_{\eta\eta} - g_\eta f_{\eta s})\} = 0, \quad (4.1)$$

and

$$g_{\eta\eta\eta} + fg_{\eta\eta} + 2r(1 - f_\eta g_\eta) + \xi\{2f_\xi g_{\eta\eta} - 2f_\eta g_{\xi\eta} + Bfg_{\eta\eta} + 2C\xi^r(gg_{\eta\eta} + 1 - g_\eta^2) - D\xi^r gg_{\eta\eta} + 2E(1 - f_\eta g_\eta) + 2A\xi^r(g_s g_{\eta\eta} - g_\eta g_{\eta s})\} = 0. \quad (4.2)$$

The expansions for f and g are as stated in (3.2). Further progress now depends on specifying the value of r, since the object here is to relate new functions to the functions defined in §3.

Consider the case $r = 1$. The boundary conditions on the various functions are as stated in (3.3), and, after the algebra of expansion of equations (4.1) and (4.2) and use of equations in the groups (3.11) and (3.13), the following results are seen to be true:

f_0 is the Blasius function, as before,

$$f_1 = B_0(f_{11} + 2f_{12}),$$

$$f_2 = B_0^2(f_{21} + 2f_{22} + 4f_{23}) + (B_1 + 2C_0 - D_0)f_{24} + 2(B_1 + D_0)f_{25} + 2(2C_0 - D_0)f_{26} + 4D_0 f_{29},$$

$$g_0 = f_0 + 2g_{11},$$

$$g_1 = f_1 + 2B_0(g_{21} + 2g_{22} - g_{23}) + 2E_0(g_{23} + 2g_{24}),$$

in which the functions f_{ij}, g_{ij} are as specified in §3.

Similar results may be derived for the case $r = 2$, and it may be noted that in each case the expansions for the velocity components are available in terms of previous results as far as terms in ξ^2. Thus, in the case $r = 2$, the functions f_0, f_1, f_2 and g_0 are obtainable.

5. Numerical solution of the equations

Equations (3.4), (3.11) and (3.13), subject to the appropriate boundary conditions, were solved numerically on the IBM 1620 computer at Bristol University, and tables of the functions f_{ij}, g_{ij} and their first and second derivatives are available on loan from the Editor of the *Journal*. As usual in computations of this type, where succeeding functions depend on functions previously calculated, numerical accuracy decreases owing to the presence of build-up error. However, f_0' is accurate to six decimal places, and it is safe to assume that f_{1j}', g_{11}' are accurate to five decimal places, and f_{2j}', g_{2j}' to four decimal places.

The shape of the velocity functions f_{ij}' and g_{ij}' is shown typically by the graph of f_{11}' in figure 1, and further information about these functions is contained in table 1. For all the functions, the first derivative becomes zero to four places of decimals when $\eta = 5{\cdot}8$.

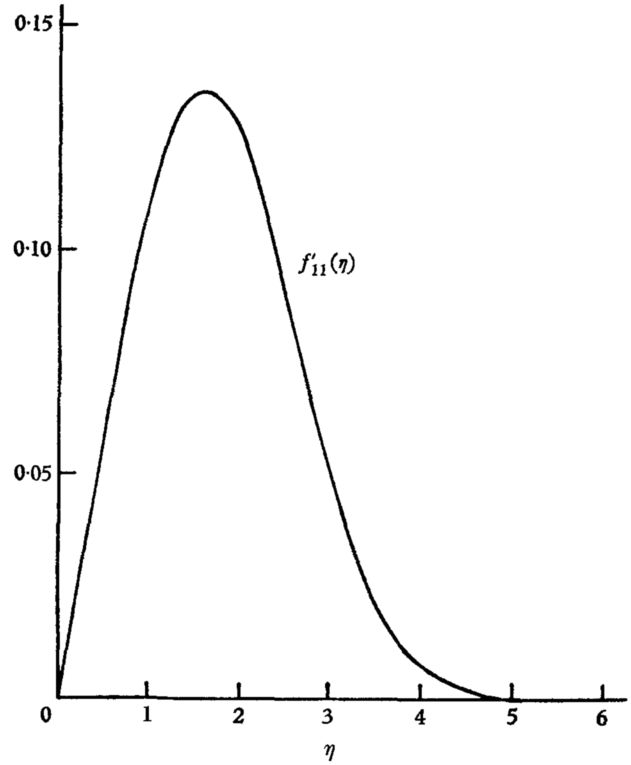

FIGURE 1. Graph of the function $f'_{11}(\eta)$ defined in § 3. For certain details of the remaining functions f_{ij}, g_{ij} see table 1.

	$X''(0)$	$X(\infty)$	Max./Min. $X'(\eta)$	η_M
f_0	0·469600	—	—	—
f_{11}	0·11740	0·30420	0·13699	1·6
f_{12}	1·03236	1·12637	0·55823	1·2
g_{11}	0·76790	0·56965	0·31928	1·0
f_{21}	0·0049	−0·0634	−0·0350	2·6
f_{22}	−0·1338	−0·6160	−0·2677	2·0
f_{23}	−0·7147	−1·6745	−0·7707	1·6
f_{24}	0·0783	0·2028	0·0913	1·6
f_{25}	0·9081	0·8976	0·4477	1·0
f_{26}	0·0641	0·1307	0·0630	1·4
f_{27}	−0·0196	−0·0507	−0·0228	1·6
f_{28}	−0·0621	−0·1144	−0·0565	1·4
f_{29}	−0·0900	−0·1628	−0·0810	1·4
g_{21}	−0·0835	−0·2602	−0·1182	1·8
g_{22}	−0·3456	−0·6877	−0·3352	1·4
g_{23}	0·6594	0·4026	0·2408	0·8
g_{24}	−0·0543	−0·0835	−0·0442	1·2

TABLE 1. Numerical values obtained from the solution of equations (3.4), (3.11) and (3.13). The column headings $X''(0)$, $X(\infty)$ denote the values of the second derivative of the function at $\eta = 0$, and the limiting value of the function as $\eta \to \infty$, respectively. Max./Min. $X'(\eta)$ refers to the maximum or minimum value of the function, as appropriate, and η_M is the value of η at which this is attained.

6. Application of results

The example mentioned in §2 serves to illustrate an application of the general results. A circular cylinder of radius a is mounted with its axis normal to the plate, and at distance na downstream from the edge of the plate, as shown in figure 2. If the main stream has uniform velocity U_c at infinity in the x-direction, the (potential) main-stream flow arising from the disturbance of the cylinder has components

$$U = U_c\left\{1+\frac{s^2-(\xi-n)^2}{\{(\xi-n)^2+s^2\}^2}\right\}, \quad V = -\frac{2U_c s(\xi-n)}{\{(\xi-n)^2+s^2\}^2}, \tag{6.1}$$

in terms of the non-dimensional co-ordinates ξ, s. These are the components of the main-stream flow over the plate, with the understanding that the region of plate under consideration is that over which the main stream is uninfluenced by boundary-layer and wake effects arising from the cylinder itself.

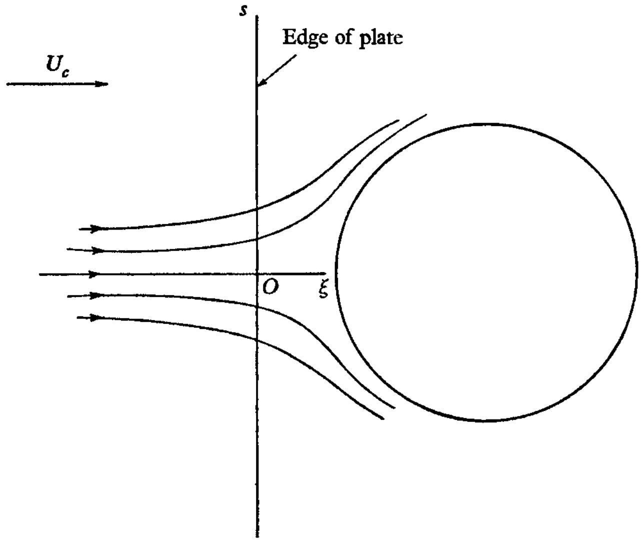

FIGURE 2. Streamlines in the main stream for flow round a circular cylinder, the cylinder being mounted with its axis normal to a flat plate and downstream from the edge of the plate.

The quantities A, B etc. may be deduced from expressions (6.1), followed by expansions in powers of ξ to derive the coefficients which are required in expressions (3.10) and (3.12). With use of table 1, the properties of the boundary layer such as shear stress components and displacement thicknesses may then be evaluated, though hereafter only the shear stress will be considered.

In three-dimensional boundary layers, a feature of special interest is the extent of the departure of direction of flow in the layer from the direction of the main stream. Reference may be made for example to Rosenhead (1963, p. 474) for some discussion of the influence of this secondary flow in the boundary layer with regard to separation effects on swept wings. The calculation of the direction of the limiting streamlines on the plate is obtained from results for the components τ_x, τ_y of the shear stress, in the directions Ox, Oy respectively.

These are

$$\left.\begin{aligned}\tau_x &= \mu\left[\frac{\partial u}{\partial z}\right]_{z=0} = \mu\left[\frac{U^3}{2\nu a\xi}\right]^{\frac{1}{2}}\left\{\frac{\partial^2 f_0}{\partial\eta^2}+\xi\frac{\partial^2 f_1}{\partial\eta^2}+\xi^2\frac{\partial^2 f_2}{\partial\eta^2}\right\}_{\eta=0} = \mu\left[\frac{U_c^3}{2\nu a\xi}\right]^{\frac{1}{2}} T_x, \text{ say,}\\ \tau_y &= \mu\left[\frac{\partial v}{\partial z}\right]_{z=0} = \mu\left[\frac{V^2U}{2\nu a\xi}\right]^{\frac{1}{2}}\left\{\frac{\partial^2 g_0}{\partial\eta^2}+\xi\frac{\partial^2 g_1}{\partial\eta^2}+\xi^2\frac{\partial^2 g_2}{\partial\eta^2}\right\}_{\eta=0} = \mu\left[\frac{U_c^3}{2\nu a\xi}\right]^{\frac{1}{2}} T_y,\end{aligned}\right\} \quad (6.2)$$

to order ξ^2. Thus at any point (ξ, s), if α is the inclination of the streamline in the main stream to Ox and $\alpha+\epsilon$ is the corresponding angle for the limiting streamline through (ξ, s),

$$\tan\epsilon = (U\tau_y - V\tau_x)/(U\tau_x + V\tau_y). \quad (6.3)$$

s	ξ	T_x	T_y	α	ϵ
0·2	0·02	0·1859	0·0495	11·8	3·1
	0·04	0·1643	0·0589	12·5	7·2
	0·06	0·1420	0·0691	13·3	12·7
	0·08	0·1191	0·0802	14·2	19·8
	0·10	0·0958	0·0922	15·1	28·8
	0·12	0·0725	0·1051	16·1	39·3
	0·14	0·0493	0·1189	17·2	50·3
0·4	0·04	0·2154	0·1080	19·6	7·0
	0·08	0·1897	0·1432	21·5	15·5
	0·12	0·1636	0·1838	23·7	24·6
	0·16	0·1375	0·2306	26·1	33·1
	0·20	0·1115	0·2842	28·8	39·8
0·6	0·04	0·2819	0·1399	21·6	4·8
	0·08	0·2770	0·1787	23·1	9·7
	0·12	0·2738	0·2225	24·8	14·3
	0·16	0·2722	0·2720	26·6	18·4
	0·20	0·2722	0·3278	28·5	20·2
0·8	0·04	0·3482	0·1530	20·7	3·1
	0·08	0·3587	0·1878	21·8	5·8
	0·12	0·3724	0·2259	23·0	8·3
	0·16	0·3891	0·2676	24·2	10·3
	0·20	0·4093	0·3133	25·4	12·0
1·0	0·04	0·4047	0·1510	18·6	1·9
	0·08	0·4237	0·1783	19·4	3·5
	0·12	0·4462	0·2073	20·1	4·8
	0·16	0·4721	0·2380	20·9	5·9
	0·20	0·5019	0·2703	21·7	6·6

TABLE 2. Values of T_x, T_y, α and ϵ for the case $n = 1{\cdot}5$ in curved flow over a flat plate due to disturbance by a cylinder (see § 6). The direction of streamlines in the main stream, α, and the deflexion of the limiting streamlines, ϵ, are measured in degrees. The quantities T_x, T_y are related to the shear stress components τ_x, τ_y by

$$\tau_x = \mu(U_c^3/2\nu a\xi)^{\frac{1}{2}}\, T_x, \quad \tau_y = \mu(U_c^3/2\nu a\xi)^{\frac{1}{2}}\, T_y.$$

Calculations were made for the case $n = 1{\cdot}5$, and table 2 shows results for T_x, T_y, α and ϵ for various values of ξ and s. Results (6.2), of course, each contains only the first three terms of an infinite series, and the question of the radius of convergence of the series arises. The coefficients in the series are functions of s, and certainly for large values of s the coefficients of $\xi^r (r \geqslant 1)$ tend to zero, and it may be expected that the series will then converge for $|\xi| \leqslant R(s)$, where $R(s)$ is large. In the limit as $s \to \infty$, $U = U_c$, $V = 0$, and τ_x is given by the first term in (6.2). For small values of s, however, the only available guidance is obtained from the numerical values of $(\partial^2 f_1/\partial\eta^2)_{\eta=0}$, etc., for various values of s. These are given in table 3, from which it seems reasonable to assume in table 2 that the calculations for the range of values of ξ shown in each case should be acceptable.

The results for $s = 0{\cdot}2$ are particularly striking, since they indicate considerable deflexion of the limiting streamlines as ξ increases, arising from pronounced decrease in the value of T_x, with increase in T_y. Indeed, T_x becomes negative at $\xi = 0{\cdot}2$, which would imply back flow in the boundary layer at this stage. The original boundary-layer approximation is then no longer valid, but it is interesting to note that experimental results for a related investigation on turbulent boundary layers by Hornung & Joubert (1963) showed regions of strong back flow upstream from the cylinder, in consequence of the rise in pressure before the cylinder.

s	$\partial^2 f_1/\partial\eta^2$	$\partial^2 f_2/\partial\eta^2$	$\partial^2 g_1/\partial\eta^2$	$\partial^2 g_2/\partial\eta^2$
0·0	−2·0771	−7·4981	4·2708	4·5394
0·2	−1·6421	−4·9648	4·0654	3·7350
0·4	−0·7841	−1·0350	3·5440	1·9875
0·6	−0·1069	0·8372	2·9037	0·4479
0·8	0·2488	1·1419	2·2952	−0·4467
1·0	0·3831	0·9291	1·7782	−0·8510
1·2	0·4035	0·6565	1·3568	−0·9981
1·4	0·3746	0·4407	1·0162	−1·0323
1·6	0·3286	0·2896	0·7395	−1·0232
1·8	0·2803	0·1881	0·5124	−1·0002
2·0	0·2356	0·1207	0·3241	−0·9749

TABLE 3. Values of coefficients in expressions (6·2), evaluated at $\eta = 0$, for the case $n = 1{\cdot}5$ in curved flow over a flat plate due to disturbance by a cylinder. The value of $(f_0'')_{\eta=0}$ is 0·4696.

For large values of s, the shear stress τ_x assumes the ordinary flat-plate value for a constant main stream, namely,

$$\tau_x = 0{\cdot}4696\,\mu(U_c^3/2\nu a\xi)^{\frac{1}{2}}$$

so that $T_x = 0{\cdot}4696 = (f_0'')_{\eta=0}$. Near the cylinder, however, T_x assumes values both in excess and defect of this, by reason of the spatial variation of velocity of the main stream. It is of interest, therefore, to calculate the influence of the cylinder on the drag of a central section of plate, of depth ξ_1 (measured from the leading edge) and of span $2s_1$, by evaluating the total excess $Q(\xi_1, s_1)$ of force over the force exerted by a uniform main stream, that is

$$Q(\xi_1, s_1) = \int_{-s_1}^{s_1}\int_0^{\xi_1} \mu\left[\frac{U_c^3 a^3}{2\nu\xi}\right]^{\frac{1}{2}} [T_x - (f_0'')_{\eta=0}]\, d\xi\, ds.$$

The integrand may be expanded in powers of ξ, and one integration leads to the result

$$Q(\xi_1, s_1) = \mu(2U_c a)^{\frac{3}{2}} (\xi_1/\nu)^{\frac{1}{2}} (m_0 + m_1 \xi_1 + m_2 \xi_1^2), \tag{6.4}$$

valid for small values of ξ_1, where the coefficients m_r are functions of s_1. The integrals which determine the m_r do not appear to respond to analytical evaluation, even in the limit $s_1 \to \infty$. They have been evaluated numerically for the case

s_1	m_0	m_1	m_2
1·0	−0·1966	−0·1378	−0·0297
5·0	−0·1078	0·1241	0·1002
10·0	−0·0471	0·1394	0·0975
15·0	−0·0247	0·1412	0·0971
20·0	−0·0132	0·1416	0·0969
25·0	−0·0062	0·1418	0·0969
30·0	−0·0016	0·1419	0·0969
lim $s_1 \to \infty$	0·0219	0·1423	0·0968

TABLE 4. Values of the coefficients m_0, m_1, m_2, in result (6·4), for the case $n = 1{\cdot}5$, in the determination of excess force on a section of plate arising from curved flow over the plate due to disturbance by a cylinder.

$n = 1{\cdot}5$, and some results are displayed in table 4. In this instance, as s_1 increases, m_0, m_1, m_2 eventually are all positive. Thus for sections of large span, namely $s_1 > 32$ approximately, Q is positive for all values of ξ_1. However, since m_0 is negative for all values of s_1 less than 32, Q is negative for sufficiently small values of ξ_1, and even for all values of ξ_1 in the range $0 < \xi_1 \leqslant 0{\cdot}1$, Q remains negative provided $s_1 < 18$.

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REFERENCES

COOKE, J. C. 1959 Approximate calculation of three-dimensional laminar boundary layers. *Aero. Res. Counc. London, R. & M.* no. 3201.

HANSEN, A. G. & HERZIG, H. Z. 1956 Cross flows in laminar incompressible boundary layers. *N.A.C.A. Tech. Note* no. 3651.

HORNUNG, H. G. & JOUBERT, P. N. 1963 Mean velocity profile in turbulent boundary layers. *J. Fluid Mech.* **15**, 368–384.

LOOS, H. G. 1955 A simple laminar boundary layer with secondary flow. *J. Aero. Sci.* **22**, 35–40.

ROSENHEAD, L. 1963 (Ed.) *Laminar Boundary Layers.* Oxford University Press.

SOWERBY, L. 1954 Secondary flow in a boundary layer. *R.A.E. Rep.* no. Aero. 2512.